2.2. The inverse of a matrix.

We are interested in defining an operation for matrices similar to division of matrices.

Since for real numbers

$$a/b = a \cdot \frac{1}{b} = a b^{-1};$$

Notice $b b^{-1} = b \frac{1}{b} = 1$, then

a good candidate would be a matrix B^{-1} for a given B, such that

$$B B^{-1} = I$$

<u>Definition</u> A $n \times n$ square matrix

<u>A is said to be invertible</u> if there is an $n \times n$ matrix C such that

$$CA = I \text{ and } AC = I \qquad (1.1)$$

where $I = I_n$, the $n \times n$ identity matrix.

The matrix C satisfying (1.1) is called the inverse of A.

Exercise #3

Consider the matrix

$$A = \begin{bmatrix} 7 & 3 \\ -6 & -3 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 1 & 1 \\ -2 & -7/3 \end{bmatrix}$$

then

$$CA = \begin{bmatrix} 1 & 1 \\ -2 & -7/3 \end{bmatrix} \begin{bmatrix} 7 & 3 \\ -6 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Similarly,

$$AC = \begin{bmatrix} 7 & 3 \\ -6 & -3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -2 & -7/3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

<u>Proposition.</u>— If the inverse of A exists, this matrix is uniquely determined.

<u>Proof.</u>— If C is an inverse of A and B is another inverse of A, then

$$B = BI = B(AC) = (BA)C = IC = C$$

Thus,

$$\boxed{B = C}$$

Therefore, if A is invertible, the inverse matrix C is unique. This unique inverse is commonly denoted as A^{-1}.

Usually, if A is not invertible then A is called singular matrix. Otherwise, A is called nonsingular matrix.

Thm 4.- Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

If $ad-bc \neq 0$, then A is invertible and

$$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \qquad (3.1)$$

If $ad-bc=0$, then A is not invertible.

Proof. We will construct A^{-1} from the definition

$$A^{-1}A = I$$

If we define $A^{-1} = \begin{bmatrix} \tilde{a} & \tilde{b} \\ \tilde{c} & \tilde{d} \end{bmatrix}$ we need to find $\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}$ in terms of a, b, c, and d.

Four equations in these four unknowns can be obtained from the equation

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I = A^{-1}A = \begin{bmatrix} \tilde{a} & \tilde{b} \\ \tilde{c} & \tilde{d} \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a\tilde{a}+c\tilde{b} & b\tilde{a}+d\tilde{b} \\ a\tilde{c}+c\tilde{d} & b\tilde{c}+d\tilde{d} \end{bmatrix}$$

Therefore,

$$\begin{cases} a\tilde{a} + c\tilde{b} + 0\tilde{c} + 0\tilde{d} = 1 \\ b\tilde{a} + d\tilde{b} + 0\tilde{c} + 0\tilde{d} = 0 \\ 0\tilde{a} + 0\tilde{b} + a\tilde{c} + c\tilde{d} = 0 \\ 0\tilde{a} + 0\tilde{b} + b\tilde{c} + d\tilde{d} = 1 \end{cases}$$

4 equations and 4 unknowns: $\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}$.

Augmented matrix:

$$\begin{bmatrix} a & c & 0 & 0 & 1 \\ b & d & 0 & 0 & 0 \\ 0 & 0 & a & c & 0 \\ 0 & 0 & b & d & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & c/a & 0 & 0 & 1/a \\ 0 & -\frac{bc}{a}+d & 0 & 0 & -b/a \\ 0 & 0 & 1 & c/a & 0 \\ 0 & 0 & 0 & -\frac{bc}{a}+d & 1 \end{bmatrix}$$

then,

$$\frac{ad-bc}{a}\tilde{d} = 1 \Rightarrow \boxed{\tilde{d} = \frac{a}{ad-bc}}, \quad ad-bc \neq 0$$

$$\tilde{c} + \frac{c}{a}\tilde{d} = 0 \Rightarrow \boxed{\tilde{c} = \frac{-c}{\cancel{a}}\left(\frac{\cancel{a}}{ad-bc}\right) = \frac{-c}{ad-bc}}$$

$$\frac{ad-bc}{\cancel{a}}\tilde{b} = -\frac{b}{\cancel{a}} \Rightarrow \boxed{\tilde{b} = \frac{-b}{ad-bc}} \quad \text{and}$$

$$\tilde{a} + \frac{c}{a}\tilde{b} = \frac{1}{a} \Rightarrow \boxed{\tilde{a} = \frac{1}{a} - \frac{c}{a}\left(\frac{-b}{ad-bc}\right) = \frac{d}{ad-bc}}$$

Finally,

$$A^{-1} = \begin{bmatrix} \tilde{a} & \tilde{b} \\ \tilde{c} & \tilde{d} \end{bmatrix} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

We can trivially show that $AA^{-1} = I$ also.

Now, we can use the previous formula (3.1) given by thm 4 to construct the inverse of the matrix $A = \begin{bmatrix} 7 & 3 \\ -6 & -3 \end{bmatrix}$ of exercise # 3.

In fact, $\det A = -21 + 18 = -3 \neq 0$

therefore,

$$A^{-1} = \frac{-1}{3} \begin{bmatrix} -3 & -3 \\ 6 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -2 & -7/3 \end{bmatrix} \checkmark$$

The quantity $ad-bc$ is called determinant of A, or

$$\det A = ad-bc$$

Another statement of thm 4 is

"A matrix $A_{2\times 2}$ is invertible if and only if $\det A \neq 0$."

Thm 5.- If $A_{n\times n}$ is invertible then, for each $\vec{b}\in\mathbb{R}^n$, the equation $A\vec{x}=\vec{b}$ has the unique solution $\vec{x}=A^{-1}\vec{b}$.

Proof.- This is an existence and uniqueness thm.

i) Existence. Pick any $\vec{b}\in\mathbb{R}^n$, then $\vec{x}=A^{-1}\vec{b}$ is a solution of $A\vec{x}=\vec{b}$, since

$$A(A^{-1}\vec{b}) = (AA^{-1})\vec{b} = I\vec{b} = \vec{b}.$$

ii) <u>Uniqueness</u>: We will prove that $\vec{x} = A^{-1}\vec{b}$ is the only solution.

If $y \in \mathbb{R}^n$ is another solution of $A\vec{x} = \vec{b}$, then

$$A\vec{y} = \vec{b}$$

then $$A^{-1}(A\vec{y}) = A^{-1}\vec{b} \Rightarrow (A^{-1}A)y = A^{-1}\vec{b}$$

$$\Rightarrow I\vec{y} = A^{-1}\vec{b} \Rightarrow \boxed{\vec{y} = A^{-1}\vec{b}}.$$

<u>Thm 6.</u>- a) $A_{n\times n}$ invertible $\Rightarrow$ A^{-1} is invertible and

$$(A^{-1})^{-1} = A.$$

b) $A_{n\times n}$, $B_{n\times n}$ both invertibles $\Rightarrow$

i) AB is also invertible, and

ii) $(AB)^{-1} = B^{-1}A^{-1}$.

c) $A_{n\times n}$ invertible $\Rightarrow$ i) A^T is also invertible and

ii) $(A^T)^{-1} = (A^{-1})^T$.

<u>Proof.</u>- a) If A is invertible then

$$AA^{-1} = I \text{ and } A^{-1}A = I \qquad (7.1)$$

By definition, equation (7.1) also implies that A^{-1} is invertible and A is its inverse.

b) To prove (b), we define

$C = B^{-1}A^{-1}$ and <u>show that</u>

<u>$(AB)C = I$ and $C(AB) = I$</u> do it!

then from the definition AB is invertible and

$$C = (AB)^{-1} = B^{-1}A^{-1}.$$

c) To prove (c), we define.

$C = (A^{-1})^T$ and show that

$$A^T(A^{-1})^T = I \quad \text{and} \quad (A^{-1})^T A^T = I.$$

Then from the definition

$$(A^{-1})^T = C = (A^T)^{-1}$$

<u>Corollary.</u>– If $A_1, A_2 .., A_q$ are invertibles $n\times n$ matrices then,

$$(A_1 A_2 \dots A_q)^{-1} = A_q^{-1} A_{q-1}^{-1} \dots A_2^{-1} A_1^{-1}.$$

Elementary Matrices. Algorithm for finding A^{-1}.

Consider the matrices

$$E_1 = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad E_3 = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$$

and

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Notice that

left multiplication $E_i A$

$$E_1 A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c+2a & d+2b \end{bmatrix}$$ Replacement of 2nd row by $2 \times$ row 1 + row 2

$$E_2 A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} c & d \\ a & b \end{bmatrix}$$ Interchange of row 2 and row 1.

$$E_3 A = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ 3c & 3d \end{bmatrix}$$ multiplication of row 2 by 3

Notice that E_1, E_2, and E_3 are obtained from the identity matrix $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ by performing a single elementary row operation.

In fact,

E_1 is obtained replacing the 2^{nd} row by $2 \times$ row 1 + row 2. on the identity matrix I_2.

E_2 is obtained by interchanging row 1 with row 2 on I_2.

E_3 is obtained from I_2 by multiplying row 2 by 3.

Proposition.- For a matrix $A_{m \times n}$ any row operation performed on A can be written as EA,

Where $E_{m \times m}$ matrix is obtained by performing the same row operation on the identity matrix, I_m

Proof.- It's a trivial extension of the 2×2 case presented above.

Notice that

$$\overset{E_1}{\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}} \overset{C}{\begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}} = \begin{bmatrix} 1 & 0 \\ 2-2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$$

Similarly, it can be proved

$$CE_1 = I_2, \text{ then } \boxed{C = E_1^{-1}}$$

<u>Thm 1.5.2</u> Every elementary matrix is invertible, and the inverse is also an elementary matrix.

<u>Idea</u>: of proof.

- Scaling:

$$\begin{bmatrix} 1/c & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} c & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

- Interchange:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- <u>Replacement</u>:

$$\begin{bmatrix} 1 & 0 & -c \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & c \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

C corresponds to the reverse operation carried out by E_1, on I_2. It means while

2^{nd} Row of E_1 consists of $2 \times$ row 1 + row 2 on I_2

then 2^{nd} Row of $C = E_1^{-1}$ consists of $-2 \times$ row 1 + row 2 on I_2.

This definition of Elementary Matrices and its effect on a matrix can be easily extended to any matrix $A_{m \times n}$ and the identity matrix I_m.

Def.- An Elementary Matrix $E_{m \times m}$ is one that is obtained by performing a single elementary row operation on an identity matrix I_m.

Thm 7.-

a) $A_{n \times n}$ is invertible if and only if A is row equivalent to I_n.

b) Any sequence of elementary row operations that reduces A to I_n, also transforms I_n into A^{-1}

Proof.- $(\Longrightarrow)$ A invertible $\Rightarrow$ $A \sim I_n$.

If A is invertible $\Rightarrow$ $A\vec{x}=\vec{0}$ has only the trivial solution $\left(\text{Since if } A\vec{x}=\vec{0} \Rightarrow A^{-1}(A\vec{x}) = A\vec{0}=\vec{0} \text{ or } (\underbrace{A^{-1}A}_{=I})\vec{x}=\vec{0} \Rightarrow \vec{x}=\vec{0}\right)$

Therefore, the pivot positions of A are along the diagonal. Otherwise, A will have an echelon form as

$$\begin{matrix} 0 & 0 & 0 & \cdots & \overset{i}{1} & * & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 1 & \cdots & 0 & 0 & 0 \end{matrix}$$

and the corresponding system of equations would have x_i and x_{i+2} as basic variables, but x_{i+1} as a free variable, contradicting that $\vec{x}=\vec{0}$ is the only solution of $A\vec{x}=\vec{0}$.

If the pivot positions of A are along the diagonal then the row-reduced echelon form of A is

$$\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & & \ddots & 0 \\ 0 & 0 & & 1 \end{pmatrix} = I_n, \quad \text{and} \quad A \sim I_n.$$

($\Leftarrow$) $A \sim I_n \Rightarrow A$ invertible.

If $A \sim I_n$ then, there exist elementary matrices $E_1, \dots, E_p$ Such that

$$A \sim E_p(E_{p-1} \dots E_1 A) = I_n.$$

or

$$(E_p E_{p-1} \dots E_1)A = I_n \qquad (13.1)$$

Notice that $(E_p E_{p-1} \dots E_1)$ is invertible because it is the product of invertible matrices. If we call the inverse of $E_p E_{p-1} \dots E_1$ as $(E_p E_{p-1} \dots E_1)^{-1}$, this is also invertible because it is the inverse of an invertible matrix.

Now, mutiplying to the left of the two sides of (13.1). by $(E_p E_{p-1} \dots E_1)^{-1}$ leads to

$$\underbrace{(E_p E_{p-1} \dots E_1)^{-1}(E_p \dots E_1)}A = (E_p E_{p-1} \dots E_1)^{-1} I_n$$

$$I_n A = (E_p E_{p-1} \dots E_1)^{-1} I_n$$

$$\Rightarrow \qquad A = (E_p E_{p-1} \dots E_1)^{-1}$$

Since $(E_p \dots E_1)^{-1}$ is invertible, A is also invertible.

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b) and

$$A^{-1} = [(E_p \dots E_1)^{-1}]^{-1} = E_p \dots E_1$$

or

$$\boxed{A^{-1} = (E_p \dots E_1) I_n}$$

Therefore, A^{-1} can be obtained by successively multiplying I_n by $E_1, E_2 .., E_p$.
This is the same sequence of operations used to reduce A to I_n.

The proof of this theorem gives a practical method to determine if a given matrix $A_{n \times n}$ has an inverse and obtain it when this exists.

Probl. 6a Sect 1.5 (10th Ed. #13)

$$[A|I] = \left[\begin{array}{ccc|ccc} 3 & 4 & -1 & 1 & 0 & 0 \\ 1 & 0 & 3 & 0 & 1 & 0 \\ 2 & 5 & -4 & 0 & 0 & 1 \end{array}\right] \sim$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 3 & 4 & -1 & 1 & 0 & 0 \\ 2 & 5 & -4 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 4 & -10 & 1 & -3 & 0 \\ 0 & 5 & -10 & 0 & -2 & 1 \end{array}\right] \sim$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & -5/2 & 1/4 & -3/4 & 0 \\ 0 & 5 & -10 & 0 & -2 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & -5/2 & 1/4 & -3/4 & 0 \\ 0 & 0 & 5/2 & -5/4 & 7/4 & 1 \end{array}\right] \sim$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & -5/2 & 1/4 & -3/4 & 0 \\ 0 & 0 & 1 & -2/4 & 7/10 & 2/5 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & -5/2 & 1/4 & -3/4 & 0 \\ 0 & 0 & 1 & -1/2 & 7/10 & 2/5 \end{array}\right] \sim$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3/2 & -11/10 & -6/5 \\ 0 & 1 & 0 & -1 & 1 & 1 \\ 0 & 0 & 1 & -1/2 & 7/10 & 2/5 \end{array}\right]$$

$$\Rightarrow A^{-1} = \begin{bmatrix} 3/2 & -11/10 & -6/5 \\ -1 & 1 & 1 \\ -1/2 & 7/10 & 2/5 \end{bmatrix}$$